Chapter 5
Quantum Phenomena in Simple Systems in Nonlinear Optics

Abstract In this chapter we will analyse some simple processes in nonlinear optics where analytic solutions are possible. This will serve to illustrate how the formalism developed in the preceding chapters may be applied. In addition, the simple examples chosen illustrate many of the quantum phenomena studied in more complex systems in later chapters.

This chapter will serve as an introduction to how quantum phenomena such as photon antibunching, squeezing and violation of certain classical inequalities may occur in nonlinear optical systems. In addition, we include an introduction to quantum limits to amplification.

5.1 Single-Mode Quantum Statistics

A single-mode field is the simplest example of a quantum field. However, a number of quantum features such as photon antibunching and squeezing may occur in a single-mode field. To illustrate these phenomena we consider the degenerate parametric amplifier which displays interesting quantum behaviour.

5.1.1 Degenerate Parametric Amplifier

One of the simplest interactions in nonlinear optics is where a photon of frequency 2ω splits into two photons each with frequency ω. This process known as parametric down conversion may occur in a medium with a second-order nonlinear susceptibility $\chi^{(2)}$. A detailed discussion on nonlinear optical interactions is left until Chap. 9.

We shall make use of the process of parametric down conversion to describe a parametric amplifier. In a parametric amplifier a signal at frequency ω is amplified by pumping a crystal with a $\chi^{(2)}$ nonlinearity at frequency 2ω. We consider a simple model where the pump mode at frequency 2ω is classical and the signal mode at frequency ω is described by the annihilation operator a. The Hamiltonian describing the interaction is

$$\mathscr{H} = \hbar\omega a^\dagger a - \mathrm{i}\hbar\frac{\chi}{2}\left(a^2\mathrm{e}^{2\mathrm{i}\omega t} - a^{\dagger 2}\mathrm{e}^{-2\mathrm{i}\omega t}\right) , \tag{5.1}$$

where χ is a constant proportional to the second-order nonlinear susceptibility and the amplitude of the pump. If we work in the interaction picture we have the time-independent Hamiltonian

$$\mathscr{H}_\mathrm{I} = -\mathrm{i}\hbar\frac{\chi}{2}\left(a^2 - a^{\dagger 2}\right) . \tag{5.2}$$

The Heisenberg equations of motion are

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\left[a, \mathscr{H}_\mathrm{I}\right] = \chi a^\dagger, \quad \frac{\mathrm{d}a^\dagger}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\left[a^\dagger, \mathscr{H}_\mathrm{I}\right] = \chi a . \tag{5.3}$$

The interaction picture can be viewed equivalently as transforming to a frame rotating at frequency ω.

These equations have the solution

$$a(t) = a(0)\cosh\chi t + a^\dagger(0)\sinh\chi t , \tag{5.4}$$

which has the form of a generator of the squeezing transformation, see (2.60). As such we expect the light produced by parametric amplification to be squeezed. This can immediately be seen by introducing the two quadrature phase amplitudes

$$X_1 = a + a^\dagger, \qquad X_2 = \frac{a - a^\dagger}{\mathrm{i}} \tag{5.5, 5.6}$$

which diagonalize (5.2 and 5.3)

$$\frac{\mathrm{d}X_1}{\mathrm{d}t} = +\chi X_1, \quad \frac{\mathrm{d}X_2}{\mathrm{d}t} = -\chi X_2. \tag{5.7, 5.8}$$

These equations demonstrate that the parametric amplifier is a phase-sensitive amplifier which amplifies one quadrature and attenuates the other:

$$X_1(t) = \mathrm{e}^{\chi t}X_1(0) , \tag{5.9}$$
$$X_2(t) = \mathrm{e}^{-\chi t}X_2(0) . \tag{5.10}$$

The parametric amplifier also reduces the noise in the X_2 quadrature and increases the noise in the X_1 quadrature. The variances $V(X_i,t)$ satisfy the relations

$$V(X_1,t) = \mathrm{e}^{2\chi t}V(X_1,0) , \tag{5.11}$$
$$V(X_2,t) = \mathrm{e}^{-2\chi t}V(X_2,0) . \tag{5.12}$$

For initial vacuum or coherent states $V(X_i,\ 0) = 1$, hence

$$V(X_1, t) = \mathrm{e}^{2\chi t}, \\ V(X_2, t) = \mathrm{e}^{-2\chi t}\ , \tag{5.13}$$

and the product of the variances satisfies the minimum uncertainty relation $V(X_1)$ $V(X_2) = 1$. Thus the deamplified quadrature has less quantum noise than the vacuum level. The amount of squeezing or noise reduction is proportional to the strength of the nonlinearity, the amplitude of the pump and the interaction time.

5.1.2 *Photon Statistics*

We shall next consider the photon statistics of the light produced by the parametric amplifier. First we analyse the light produced from an initial vacuum state. The intensity correlation function $g^{(2)}(0)$ in this case is

$$g^{(2)}(0) = \frac{\langle a^\dagger(t)\, a^\dagger(t)\, a(t)\, a(t)\rangle}{\langle a^\dagger(t)\, a(t)\rangle^2} \\ = 1 + \frac{\cosh 2\chi t}{\sinh^2 \chi t}\ . \tag{5.14}$$

This indicates that the squeezed light generated from an initial vacuum exhibits photon bunching ($g^{(2)}(0) > 1$). This is expected for a squeezed vacuum which must contain correlated pairs of photons.

For an initial coherent state $|\alpha\rangle$ we find the mean photon number

$$\langle a^\dagger(t)\, a(t)\rangle = |\alpha|^2 (\cosh 2\chi t + \cos 2\theta \sinh 2\chi t) + \sinh^2 \chi t\ , \tag{5.15}$$

where we have used $\alpha = |\alpha|\mathrm{e}^{\mathrm{i}\theta}$, and the intensity correlation function

$$g^{(2)}(0) \approx 1 + \frac{1}{|\alpha|^2 \mathrm{e}^{-2\chi t}} \left(\mathrm{e}^{-2\chi t} - 1\right), \quad \theta = \frac{\pi}{4}\ , \tag{5.16}$$

where $|\alpha|^2$ is large compared with $\sinh^2 \chi t$ and $\sinh \chi t\ \cosh \chi t$.

Thus under these conditions the photon statistics of the output light is antibunched. We see that a parametric amplifier evolving from an initial coherent state $|\mathrm{i}|\alpha|\rangle$ evolves towards an amplitude squeezed state with a coherent amplitude of $|\alpha|\mathrm{e}^{-\chi t}$. This reduction in amplitude is due to the dynamic contraction in the X_2 direction described by (5.10) (Fig. 5.1).

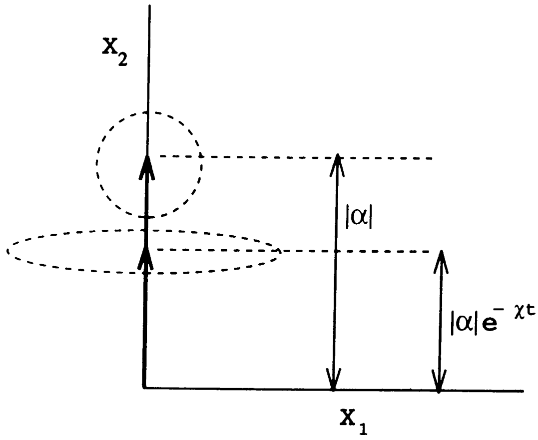

Fig. 5.1 Schematic representation of the evolution of an initial coherent state undergoing parametric amplification

5.1.3 Wigner Function

The full photon statistics of the light generated in parametric amplification may be calculated via a quasi-probability distribution. While we could choose to calculate the P function we would find that it would become singular due to the quantum correlations which build up during the amplification process. Therefore we shall calculate the Wigner distribution which is a nonsingular positive function for this problem.

The Wigner function describing the state of the parametric oscillator at any time t may now be calculated via the symmetrically ordered characteristic function,

$$\chi(\eta,t) = \mathrm{Tr}\left\{\rho(0)\,\mathrm{e}^{\eta a^\dagger(t) - \eta^* a(t)}\right\} . \tag{5.17}$$

Let us take the initial state to be the coherent state $\rho(0) = |\alpha_0\rangle\langle\alpha_0|$. Then substituting (5.4) into (5.17) we find

$$\chi(\eta,t) = \exp\left[\eta\alpha_0^*(t) - \eta^*\alpha_0(t) - \frac{|\eta|^2}{2}\cosh 2\chi t + \tfrac{1}{4}\left(\eta^2 + \eta^{*2}\right)\sinh 2\chi t\right] , \tag{5.18}$$

where

$$\alpha_0(t) = \alpha_0 \cosh\chi t + \alpha_0^* \sinh\chi t . \tag{5.19}$$

This may be written as

$$\chi(\eta,t) = \exp\left[\boldsymbol{\eta}^{\mathrm{T}} \cdot \boldsymbol{\alpha}_0^*(t) + \tfrac{1}{2}\boldsymbol{\eta}^{\mathrm{T}}\Lambda\boldsymbol{\eta}\right] , \tag{5.20}$$

$$\boldsymbol{\eta}^{\mathrm{T}} = (\eta, -\eta^*) , \tag{5.21}$$

$$\boldsymbol{\alpha}_0^{\mathrm{T}}(t) = (\alpha_0(t), \alpha_0^*(t)) \tag{5.22}$$

and

$$\Lambda = \frac{1}{2}\begin{pmatrix} \sinh 2\chi t & \cosh 2\chi t \\ \cosh 2\chi t & \sinh 2\chi t \end{pmatrix} . \tag{5.23}$$

The Wigner function is then given by the Fourier transform of $\chi(\eta, t)$, see (4.33). Using (4.36) the result is

$$W(\alpha,t) = \frac{2}{\pi}\exp\left\{\frac{1}{2}\left[\alpha - \alpha_0(t)\right]^{\mathrm{T}} C_\alpha^{-1}\left[\alpha - \alpha_0(t)\right]\right\} , \tag{5.24}$$

where $\alpha^{\mathrm{T}} = (\alpha,\ \alpha^*)$. This is a two variable Gaussian with mean $\alpha_0(t)$ and covariance matrix $C_\alpha = \Lambda$. In terms of the real variables $x_1 = \alpha + \alpha^*$. $x_2 = -\mathrm{i}(\alpha - \alpha^*)$ (corresponding to the quadrature phase operators), the Wigner function becomes

$$W(x_1,x_2) = \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\left[\boldsymbol{x} - \boldsymbol{x}_0(t)\right] C_x^{-1}\left[\boldsymbol{x} - \boldsymbol{x}_0(t)\right]\right\} , \tag{5.25}$$

where

$$C\boldsymbol{x} = \begin{pmatrix} \mathrm{e}^{2\chi t} & 0 \\ 0 & \mathrm{e}^{-2\chi t} \end{pmatrix} . \tag{5.26}$$

Thus the Wigner function is a two-dimensional Gaussian with the variance in fluctuations in the quadratures X_1 and X_2 given by the major and minor axes of the elliptic contours.

5.2 Two-Mode Quantum Correlations

In two-mode systems there are a richer variety of quantum phenomena since there exists the possibility of quantum correlations between the modes. These correlations may give rise to two mode squeezing such, as described by (2.85). There may also exist intensity and phase correlations between the modes. A simple system which displays many of the above features is the non-degenerate parametric amplifier [1].

5.2.1 Non-degenerate Parametric Amplifier

The non-degenerate parametric amplifier is a simple generalization of the degenerate parametric amplifier considered in the previous section. In this case the classical pump mode at frequency $2\omega_1$ interacts in a nonlinear optical medium with two modes at frequency ω_1 and ω_2. These frequencies sum to the pump frequency, $2\omega = \omega_1 + \omega_2$. It is conventional to designate one mode as the signal and the other as the idler.

The Hamiltonian describing this system is

$$\mathscr{H} = \hbar\omega_1 a_1^\dagger a_1 + \hbar\omega_2 a_2^\dagger a_2 + \mathrm{i}\hbar\chi\left(a_1^\dagger a_2^\dagger \mathrm{e}^{-2\mathrm{i}\omega t} - a_1 a_2 \mathrm{e}^{2\mathrm{i}\omega t}\right) , \tag{5.27}$$

where $a_1(a_2)$ is the annihilation operator for the signal (idler) mode. The coupling constant χ is proportional to the second-order susceptibility of the medium and to the amplitude of the pump.

The Heisenberg equations of motion in the interaction picture are

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = \chi a_2^\dagger \,, \tag{5.28}$$

$$\frac{\mathrm{d}a_2^\dagger}{\mathrm{d}t} = \chi a_1 \,. \tag{5.29}$$

The solutions to these equations are

$$a_1(t) = a_1 \cosh \chi t + a_2^\dagger \sinh \chi t \,, \tag{5.30}$$

$$a_2(t) = a_2 \cosh \chi t + a_1^\dagger \sinh \chi t \,. \tag{5.31}$$

If the system starts in an initial coherent stat $|\alpha_1\rangle$, $|\alpha_2\rangle$, the mean photon number in mode one after time t is

$$\begin{aligned}\langle n_1(t)\rangle &= \langle \alpha_1, \alpha_2 \left| a_1^\dagger(t) a_1(t) \right| \alpha_1, \alpha_2\rangle \\ &= |\alpha_1 \cosh \chi t + \alpha_2^* \sinh \chi t|^2 + \sinh^2 \chi t \,. \end{aligned} \tag{5.32}$$

The last term in this equation represents the amplification of vacuum fluctuations since if the system initially starts in the vacuum $(\alpha_1 = \alpha_2 = 0)$ a number of photons given by $\sinh^2 \chi t$ will be generated after a time t.

The intensity correlation functions of this system exhibit interesting quantum features. With a two-mode system we may consider cross correlations between the two modes. We shall show that quantum correlations may exist which violate classical inequalities.

Consider the moment $\langle a_1^\dagger a_1 a_2^\dagger a_2\rangle$. We may express this moment in terms of the Glauber–Sudarshan P function as follows:

$$\langle a_1^\dagger a_1 a_2^\dagger a_2\rangle = \int \mathrm{d}^2\alpha_1 \int \mathrm{d}^2\alpha_2 \, |\alpha_1|^2 \, |\alpha_2|^2 \, P(\alpha_1, \alpha_2) \,. \tag{5.33}$$

If a positive P function exists the right-hand side of this equation is the classical intensity correlation function for two fields with the fluctuating complex amplitudes α_1 and α_2. It follows from the Hölder inequality that

$$\begin{aligned}\int \mathrm{d}^2\alpha_1 \mathrm{d}^2\alpha_2 \, |\alpha_1|^2 \, |\alpha_2|^2 \, P(\alpha_1, \alpha_2) \leq & \left[\int \mathrm{d}^2\alpha_1 \mathrm{d}^2\alpha_2 \, |\alpha_1|^4 \, P(\alpha_1, \alpha_2)\right]^{1/2} \\ & \times \left[\int \mathrm{d}^2\alpha_1 \mathrm{d}^2\alpha_2 \, |\alpha_2|^4 \, P(\alpha_1, \alpha_2)\right]^{1/2} \,. \end{aligned} \tag{5.34}$$

Re-expressed in terms of operators this inequality implies

$$\langle a_1^\dagger a_1 a_2^\dagger a_2\rangle \leq [\langle (a_1^\dagger)^2 a_1^2\rangle \langle (a_2^\dagger)^2 a_2^2\rangle]^{1/2} \,, \tag{5.35}$$

a result known as the *Cauchy–Schwarz inequality*. If the two modes are symmetric as for the non-degenerate parametric amplifier this inequality implies

$$\langle a_1^\dagger a_1 a_2^\dagger a_2 \rangle \le \langle (a_1^\dagger)^2 a_1^2 \rangle \,. \tag{5.36}$$

Because we have assumed a positive P function this is a weak inequality and there exists certain quantum fields which will violate it.

It is more usual to express the Cauchy–Schwarz inequality in terms of the second-order intensity correlation functions defined for a single-mode field in (3.63). The two-mode intensity correlation function is defined by

$$g_{12}^{(2)}(0) = \frac{\langle a_1^\dagger a_1 a_2^\dagger a_2 \rangle}{\langle a_1^\dagger a_1 \rangle \langle a_2^\dagger a_2 \rangle} \,. \tag{5.37}$$

This definition together with

$$g_i^{(2)}(0) = \frac{\langle a_i^\dagger a_i^\dagger a_i a_i \rangle}{\langle a_i^\dagger a_i \rangle^2} \tag{5.38}$$

enables one to write the Cauchy–Schwarz inequality as

$$[g_{12}^{(2)}(0)]^2 \le g_1^{(2)}(0) g_2^{(2)}(0) \,. \tag{5.39}$$

A stronger inequality may be derived for quantum fields when a Glauber–Sudarshan P representation does not exist. The appropriate inequality for two non-commuting operators is, see (3.26),

$$\langle a_1^\dagger a_1 a_2^\dagger a_2 \rangle^2 \le \langle (a_1^\dagger a_1)^2 \rangle \langle (a_2^\dagger a_2)^2 \rangle \,. \tag{5.40}$$

For symmetrical systems this implies

$$\langle a_1^\dagger a_1 a_2^\dagger a_2 \rangle \le \langle (a_1^\dagger)^2 a_1^2 \rangle + \langle a_1^\dagger a_1 \rangle \tag{5.41}$$

or

$$g_{12}^{(2)}(0) \le g_1^{(2)}(0) + \frac{1}{\langle a_1^\dagger a_1 \rangle} \,. \tag{5.42}$$

We now show that the non-degenerate parametric amplifier if initially in the ground state leads to a maximum violation of the Cauchy–Schwarz inequality (5.39), as is consistent with the inequality (5.42). That is, the correlations built up in the parametric amplifier are the maximum allowed by quantum mechanics.

In this system the following conservation law is easily seen to hold,

$$n_1(t) - n_2(t) = n_1(0) - n_2(0) \,, \tag{5.43}$$

where $n_i(t) \equiv a_i^\dagger(t) a_i(t)$. This conservation law has been exploited to give squeezing in the photon number difference in a parametric amplifier as will be described in

Chap. 8. Using this relation the intensity correlation function may be written

$$\langle n_1(t)\, n_2(t)\rangle = \langle n_1(t)^2\rangle + \langle n_1(t)\,[n_2(0) - n_1(0)]\rangle\,. \tag{5.44}$$

If the system is initially in the vacuum state the last term is zero, thus

$$\langle n_1(t)\, n_2(t)\rangle = \langle a_1^\dagger(t)\, a_1^\dagger(t)\, a_1(t)\, a_1(t)\rangle + \langle a_1^\dagger(t)\, a_1(t)\rangle\,, \tag{5.45}$$

which corresponds to the maximum violation of the Cauchy–Schwarz inequality allowed by quantum mechanics.

Thus the non-degenerate parametric amplifier exhibits quantum mechanical correlations which violate certain classical inequalities. These quantum correlations may be further exploited to give squeezing and states similar to those discussed in the EPR paradox, as will be described in the following subsections.

5.2.2 *Squeezing*

In the interaction picture, the unitary operator for time evolution of the non-degenerate parametric amplifier is

$$U(t) = \exp\left[\chi t\left(a_1^\dagger a_2 - a_1 a_2\right)\right] \tag{5.46}$$

Comparison with (2.87) shows that $U(t)$ is the unitary two-mode squeezing operator, $S(G)$ with $G = -\chi t$. We will define the squeezing with respect to the quadrature phase amplitudes of the field at the local oscillator frequency ω_{LO} and phase reference θ [2].

To see this explicitly consider the positive frequency components of the field that results for a super-position of the signal and a local oscillator field in a coherent state with $\langle E_{LO}^{(-)}(t)\rangle = |\alpha|\mathrm{e}^{-\mathrm{i}(\omega_{\mathrm{LO}}t+\theta)}$. The positive frequency components of the superposed field is then well approximated by

$$E_T^{(-)}(t) = E^{(-)}(t) + |\alpha|\,\mathrm{e}^{-\mathrm{i}(\omega_{\mathrm{LO}}t+\theta)} \tag{5.47}$$

The photo-current when such a field is directed to a detector is then proportional to $i(t) \propto \langle E_T^{(-)}(t) E_T^{(+)}(t)\rangle$. We now define the average homodyne detection signal by subtracting off the known local oscillator intensity and normalising by $|\alpha|$,

$$s(t) = \langle E^{(-)}(t)\,\mathrm{e}^{\mathrm{i}(\theta+\omega_{LO}t) + E^{(+)}(t)\mathrm{e}^{-\mathrm{i}(\theta+\omega_{LO}t)}}\rangle \tag{5.48}$$

The noise in the signal will then be determined by the variance in the operator $\hat{s}(t) = E^{(-)}(t)\mathrm{e}^{\mathrm{i}(\theta+\omega_{\mathrm{LO}}t)} + E^{(+)}(t)\mathrm{e}^{-\mathrm{i}(\theta+\omega_{\mathrm{LO}}t)}$.

We can now make a change of variable for the frequencies of the signal and idler fields by writing $\omega_1 = \omega - \varepsilon$, $\omega_2 = \omega + \varepsilon$ with $\varepsilon > 0$. This change of variable anticipates a homodyne detection scheme. If we mix these two modes with a local oscillator at half the pump frequency, i.e. at $\omega_{LO} = \omega$, the resulting signal will have Fourier components at frequencies $\pm\varepsilon$.

In the new frequency variables, the total field of the signal plus idler in the Heisenberg picture is the sum of two modes $\omega \pm \varepsilon$ symmetrically displaced about the local oscillator,

$$E(t) = \frac{1}{\sqrt{2}}\left[a_1(t)\,\mathrm{e}^{-\mathrm{i}(\omega+\varepsilon)t} + a_2(t)\,\mathrm{e}^{-\mathrm{i}(\omega+\varepsilon)t} + \text{h.c.}\right]$$

where $a_i(t)$ is the solution given in (5.30 and 5.31), h.c means hermitian conjugate, and the factor $1/\sqrt{2}$ has been inserted to give a convenient definition of the vacuum. This may be written as

$$E(t) = X_\theta(t,\varepsilon)\cos(\omega t + \theta) - X_{\theta+\pi/2}(t,\varepsilon)\sin(\omega t + \theta) \tag{5.49}$$

and the quadrature phase operators are defined as

$$X_\theta(t,\varepsilon) = \frac{1}{\sqrt{2}}\left[\left(a_1(t)\,\mathrm{e}^{\mathrm{i}\theta} + a_2^\dagger(t)\,\mathrm{e}^{-\mathrm{i}\theta}\right)\mathrm{e}^{\mathrm{i}\varepsilon t} + \text{h.c.}\right]$$

$$X_{\theta+\pi/2}(t,\varepsilon) = \frac{i}{\sqrt{2}}\left[\left(a_1(t)\,\mathrm{e}^{\mathrm{i}\theta} - a_2^\dagger(t)\,\mathrm{e}^{-\mathrm{i}\theta}\right)\mathrm{e}^{\mathrm{i}\varepsilon t} + \text{h.c.}\right]$$

In this form, as $\varepsilon > 0$, we can distinguish the positive and negative frequency components of the quadrature phase operators with respect to the local oscillator frequency.

If the system starts in the vacuum state, the homodyne detection signal at $\varepsilon = 0$ (the DC signal) will have a variance given by

$$X_\theta(t,\varepsilon = 0) = \cosh 2\chi t + \cos 2\theta \sinh 2\chi t \tag{5.50}$$

Thus for $\theta = 0$, we find that

$$V\left(X_0(t,\varepsilon = 0)\right) = e^{2\chi t} \tag{5.51}$$

$$V\left(X_{\pi/2}(t,\varepsilon = 0)\right) = e^{-2\chi t} \tag{5.52}$$

Changing the phase of the local oscillator by $\pi/2$ enables one to move from enhanced to diminished noise in the homodyne signal. We note that the squeezing in the non-degenerate parametric amplifier is due to the development of quantum correlations between the signal and idler mode. The individual signal and idler modes are not squeezed as is easily verified.

5.2.3 Quadrature Correlations and the Einstein–Podolsky–Rosen Paradox

The non-degenerate parametric amplifier can also be used to prepare states similar to those discussed in the Einstein–Podolsky–Rosen (EPR) paradox [3]. In the original treatment two systems are prepared in a correlated state. One of two canonically conjugate variables is measured on one system and the correlation is such that the value for a physical variable in the second system may be inferred with certainty.

To see how this behaviour is manifested in the non-degenerate parametric amplifier we first define two sets, one for each mode, of canonically conjugate variables, i.e.,

$$X_i^\theta = a_i \mathrm{e}^{\mathrm{i}\theta} + a_i^\dagger \mathrm{e}^{-\mathrm{i}\theta} \quad (i=1,2) \ . \tag{5.53}$$

The variables X_i^θ and $X_i^{\theta+\pi/2}$ obey the commutation relation

$$\left[X_i^\theta, X_i^{\theta+\pi/2}\right] = -2\mathrm{i} \tag{5.54}$$

and are thus directly analogous to the position and momentum operators discussed in the original EPR paper.

To measure the degree of correlation between the two modes in terms of these operators, we consider the quantity

$$V(\theta,\phi) \equiv \tfrac{1}{2}\langle (X_1^\theta - X_2^\phi)^2 \rangle \ . \tag{5.55}$$

If $V(\theta,\phi)=0$ then X_1^θ is perfectly correlated with X_2^ϕ. This means a measurement of X_1^θ can be used to infer a value of X_2^ϕ with certainty. To appreciate why such a correlation should occur in the non-degenerate parametric amplifier, we can write the interaction Hamiltonian directly in terms of the defined canonical variables,

$$\begin{aligned}\mathscr{H}_1 = &-2\hbar\chi \sin(\theta+\phi)\left(X_1^\theta X_2^\phi - X_1^{\theta+\pi/2} X_2^{\phi+\pi/2}\right)\\ &-2\hbar\chi \cos(\theta+\phi)\left(X_1^{\theta+\pi/2} X_2^\phi + X_1^\theta X_2^{\phi+\pi/2}\right) \ .\end{aligned} \tag{5.56}$$

The Heisenberg equation of motion for X_1^θ is then

$$\dot{X}_1^\theta = -4\chi\left[X_2^\phi \cos(\theta+\phi) - X_2^{\phi+\pi/2} \sin(\theta+\phi)\right] \tag{5.57}$$

and we see that X_1^θ is coupled solely to X_2^ϕ when $\theta+\phi=0$.

Direct calculation of $V(\theta,\phi)$ using the solutions in (5.30 and 5.31) gives

$$V(\theta,\phi) = \cosh 2\chi t - \sinh 2\chi t \cos(\theta+\phi) \ . \tag{5.58}$$

When $\theta+\phi=0$, $V(\theta,\phi)=\mathrm{e}^{-2\chi t}$ and, for long times, $V(\theta,\phi)$ becomes increasingly small reflecting the build up of correlation between the two variables for this case. Initially, of course, the two systems are uncorrelated and $V(\theta,\phi) = 1$. As $V(\theta,\phi)$ tends to zero the system becomes correlated in the sense of the EPR paradox. As time proceeds a measurement of X_1^θ yields an increasingly certain value of X_2^ϕ. However one could equally have measured $X_1^{\theta-\pi/2}$. Thus certain values for two noncommuting observables X_2^ϕ, $X_2^{\phi+\pi/2}$ may be obtained without in anyway disturbing system 2. This outcome constitutes the centre of the *EPR argument*.

Of course, in reality no measurement enables a perfect inference to be made. To quantify the extent of the apparent paradox, we can define the variances $V_{\text{inf}}(X_2^\phi)$ and $V_{\text{inf}}(X_2^{\phi+\pi/2})$ which determine the error in inferring X_2^ϕ and $X_1^{\phi+\pi/2}$ from direct measurements on X_2^θ and $X_1^{\theta-\pi/2}$. In the case of direct measurements made on $(X_2^\phi,\, X_2^{\phi+\pi/2})$ quantum mechanics would suggest

$$V\left(X_2^\phi\right) V\left(X_2^{\phi+\pi/2}\right) \geq 4\,.$$

However the variances in the inferred values are not constrained. Thus whenever $V_{\text{inf}}(X_2^\phi)V_{\text{inf}}(X_2^{\phi+\pi/2}) < 4$, we can claim an EPR correlation paradoxically less than expected by direct measurement on the same state. This result seems to contradict the uncertainty principle. That this is not the case is seen as follows. In the standard uncertainty principle the variances are calculated with respect to the same state. However in the inference uncertainty product the variances are not calculated in the same state. That is to say $V_{\text{inf}}(X_2^{\phi+\pi/2})$ is calculated on the conditional state given a result for a measurement of $X_2^{\phi+\pi/2}$, however $V_{\text{inf}}(X_2^\phi)$ is calculated on the different conditional state given a result for the measurement of X_2^ϕ.

Ou et al. [4] performed an experimental test of these for the parametric amplifier. Using their quadrature normalization, the inferred variances indicate a paradoxical result if

$$V_{\text{inf}}\left(X_1^\phi\right) V_{\text{inf}}\left(X_1^{\phi+\pi/2}\right) \leq 2\,.$$

The experimental result for the lowest value of the product was 0.7 ± 0.01.

5.2.4 Wigner Function

The full quantum correlations present in the parametric amplifier may be represented using a quasi-probability distribution. If both modes of the amplifier are initially in the vacuum state no Glauber P function for the total system exists at any time. However, a Wigner function may be obtained. We shall proceed to derive the Wigner function for the parametric amplifier.

We may define a two mode characteristic function by a simple generalization of the single-mode definition. For both modes initially in the vacuum state this may be expressed as

$$\begin{aligned}\chi(\eta_2,\eta_2,t) &= \langle 0|\exp\left[\eta_1 a_1^\dagger(t)-\eta_1^* a_1(t)\right]\exp\left[\eta_2 a_2^\dagger(t)-\eta_2^* a_2(t)\right]|0\rangle \\ &= \exp\left[-\frac{1}{2}\left[\eta_1(t)|^2-\frac{1}{2}\left[\eta_2(t)|^2\right]\right.\right. . \end{aligned} \tag{5.59}$$

where

$$\begin{aligned}\eta_1(t) &= \eta_1\cosh\chi t-\eta_2^*\sinh\chi t, \\ \eta_2(t) &= \eta_2\cosh\chi t-\eta_1^*\sinh\chi t,\end{aligned}$$

The Wigner function is then given by

$$\begin{aligned}W(\alpha_1,\alpha_2,t) &= \frac{1}{\pi^4}\int d^2\eta_1\int d^2\eta_2\exp(\eta_1^*\alpha_1-\eta_1\alpha_1^*)\exp(\eta_2^*\alpha_2-\eta_2\alpha_2^*)\chi(\eta_1,\eta_2,t) \\ &= \frac{4}{\pi^2}\exp\left(-2|\alpha_1\cosh\chi t-\alpha_2^*\sinh\chi t|^2\right. \\ &\quad \left.-2|\alpha_2\cosh\chi t-\alpha_1^*\sinh\chi t|^2\right) .\end{aligned} \tag{5.60}$$

This distribution may be written in terms of the uncoupled c-number variables

$$\begin{aligned}\gamma_1 &= \alpha_1+\alpha_2^*, \\ \gamma_2 &= \alpha_1-\alpha_2^*.\end{aligned}$$

In these new variables the Wigner function is

$$W(\gamma_1,\gamma_2) = \frac{4}{\pi^2}\exp\left[-\frac{1}{2}\left(\frac{|\gamma_1|^2}{e^{2\chi t}}+\frac{|\gamma_2|^2}{e^{-2\chi t}}\right)\right] , \tag{5.61}$$

in which form it is particularly easy to see that squeezing occurs in a linear combination of the two modes. The variances in the two quadratures being given by $e^{-2\chi t}$ and $e^{2\chi t}$, respectively. It is interesting to note that even though the state produced contains non-classical correlations the Wigner function always remains positive.

5.2.5 Reduced Density Operator

When a two component system is in a pure state the reduced state of each component system, determined by a partial trace operation, will be a mixed state. An interesting feature of the non-degenerate parametric amplifier is that the reduced state of each mode is a thermal state, if each mode starts from the vacuum.

To demonstrate this result we first show the high degree of correlation between the photon number in each mode. The state of the total system at time t is

$$|\psi(t)\rangle = \exp\left[\chi t\left(a_1^\dagger a_2^\dagger - a_1 a_2\right)\right]|0\rangle \tag{5.62}$$

We now make use of the disentangling theorem [5]

$$e^{\theta\left(a_1^\dagger a_2^\dagger - a_1 a_2\right)} = e^{\Gamma a_1^\dagger a_2^\dagger} e^{-g\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)} e^{-\Gamma a_1 a_2} , \tag{5.63}$$

where

$$\Gamma = \tanh\,\theta,$$
$$g = \ln(\cosh\,\theta).$$

Thus

$$|\psi(t)\rangle = e^{-g} e^{\Gamma a_1^\dagger a_2^\dagger}|0\rangle = (\cosh\chi t)^{-1} \sum_{n=1}^{\infty} (\tanh\chi t)^n\,|n,n\rangle , \tag{5.64}$$

where $|n,n\rangle \equiv |n\rangle_1 \bigotimes |n\rangle_2$. As photons are created in pairs there is perfect correlation between the photon number in each mode. The reduced state of either mode is then easily seen to be

$$\rho_{\mathrm{i}}(t) = \mathrm{T}r_j\left\{|\Psi(t)\rangle\langle\Psi(t)|\right\} = (\cosh\chi t)^{-2} \sum_{n=0}^{\infty} (\tanh\chi t)^{2n}\,|n\rangle i\langle n| \quad i \neq j\,. \tag{5.65}$$

This is a thermal state with mean $\bar{n} = \sinh^2\chi t$, having strong analogies with the Hawking effect associated with the thermal evaporation of black holes.

Suppose, however, that a photodetector with quantum efficiency μ has counted m photons in mode b. What is the state of mode a conditioned on this result? Such a conditional state for mode a is referred to as the *selected state* as it is selected from an ensemble of systems each with different values for the number of photons counted in mode b. We shall now describe how the conditional state of mode a may be calculated.

In Chap. 3 we saw that the probability to detect m photons from a field with the photon-number distribution $P(n)$ and detector efficiency μ is

$$P_\mu(m) = \sum_{n=m}^{\infty} \binom{n}{m} (1-\mu)^{n-m}\,\mu^m P_1(n) , \tag{5.66}$$

where $P_1(n)$ is the photon number distribution for the field. This equation may be written as

$$P_\mu(m) \approx \mathrm{Tr}\left\{\rho\Upsilon_\mu^\dagger(m)\Upsilon_\mu(m)\right\} \tag{5.67}$$

and the operator Υ on mode b is defined by

$$\Upsilon_\mu(m) = \sum_{n=m}^{\infty} \binom{n}{m}^{1/2} (1-\mu)^{(n-m)/2}\,\mu^{m/2}|n-m\rangle_b\langle n|\,. \tag{5.68}$$

Note that when $\mu \to 1$ this operator approaches the projection operator $|0\rangle_b\langle m|$. This is quite different to the projection operator $|m\rangle_b\langle m|$ that a naive application of the von Neumann projection postulate would indicate for photon counting measurements, and reflects the fact that real photon-counting measurements are destructive,

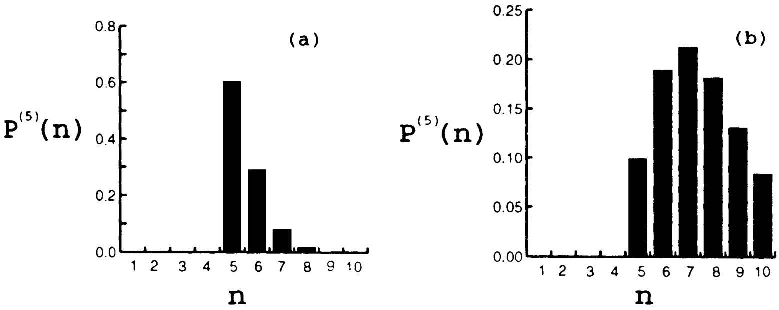

Fig. 5.2 Photon number distribution for mode a given that 5 photons are counted in mode b of a parametric amplifier. (**a**) $\mu = 0.9$. (**b**) $\mu = 0.6$

i.e. photons are absorbed upon detection. The conditional state of mode a is then given by

$$\rho^{(m)} = \left(P_\mu(m)\right)^{-1} \mathrm{Tr}_b \left\{ \Upsilon_\mu(m)_\rho \Upsilon_m^\dagger(m) \right\} . \tag{5.69}$$

This equation is a generalisation of the usual projection postulate. In the case of the correlated two mode state in (5.65), the conditional state of mode a becomes

$$\rho^{(m)} = \left(P_\mu(m)\right)^{-1} \sum_{n=m}^{\infty} \binom{n}{m} \mu^m (1-\mu)^{n-m} |n\rangle\langle n| , \tag{5.70}$$

with

$$P_\mu(m) = (1+\bar{n})^{-1} (\lambda\mu)^m [1 - \lambda(1-\mu)]^{(m+1)} , \tag{5.71}$$

where $\lambda = \tanh^2 \chi t$, $\bar{n} = \sinh^2 \chi t$. Equation (5.70) represents a state with *at least* m quanta. In Fig. 5.2 we show the photon number distribution $P^{(m)}(n)$ for this conditional state. As one would expect, when $\mu \to 1$, this approaches a number state $|m\rangle$. It should be noted, however, that the conditional state computed above refers to a situation in which the counting is done *after* the interaction which produces the correlated state, is turned off. In a cavity configuration, however, it is likely that photon counting is proceeding simultaneously with the process of parametric amplification. In that case one must proceed a little differently, however the overall result is much the same, i.e., mode a is left with at least m quanta. The details of this more complicated calculation will be found in the paper by *Holmes* et al. [6].

5.3 Quantum Limits to Amplification

The non-degenerate parametric amplifier exemplifies many features of general linear amplification. One such feature is the limit placed on the amplifier gain if the output is to be squeezed. To see how this limit arises, and to see how it might be overcome, we write the solutions (5.30 and 5.31) in the form

$$X_{1,\mathrm{OUT}}^{\theta} = G^{1/2} X_{1,\mathrm{IN}}^{\theta} + (G-1)^{1/2} X_{2,\mathrm{IN}}^{\theta} \,, \tag{5.72}$$

where $G = \cosh^2 \chi t$ and the quadrature phase operators are defined in (5.53). The subscript IN denotes operators defined at $t = 0$ and the subscripts 1 and 2 refer to the signal and idler modes, respectively. In (5.72) the first term describes the amplification of the quadrature and the second term the noise added by the amplifier. The variances obey the equation

$$V\left(X_{1,\mathrm{OUT}}^{\theta}\right) = GV\left(X_{1,\mathrm{IN}}^{\theta}\right) + (G-1)V\left(X_{2,\mathrm{IN}}^{\theta}\right) \,. \tag{5.73}$$

The maximum gain consistent with any squeezing at the output is

$$G_{\mathrm{MAX}} = \frac{1 + V\left(X_{2,\mathrm{IN}}^{\theta}\right)}{V\left(X_{1,\mathrm{IN}}^{\theta}\right) + V\left(X_{2,\mathrm{IN}}^{\theta}\right)} \,. \tag{5.74}$$

If the idler mode is in the vacuum state, $V(X_{2,\mathrm{IN}}^{\theta}) = 1$ then

$$G_{\mathrm{MAX}} = \frac{2}{1 + V\left(X_{1,\mathrm{IN}}^{\theta}\right)} \,, \tag{5.75}$$

which gives a maximum gain of 2 for a highly squeezed state at the signal input. For higher values of the gain the squeezing at the signal output is lost due to contamination from the amplification of vacuum fluctuation in the idler input.

Greater gains may be achieved while still retaining the squeezing in the output signal if the input to the idler mode, is squeezed $(V(X_{1,\mathrm{IN}}^{\theta}) < 1)$.

If we define the total noise in the signal as the sum of the noise in the two quadratures

$$N = \mathrm{Var}\left\{X_1^{\theta}\right\} + \mathrm{Var}\left\{X_1^{\theta+\pi/2}\right\} \tag{5.76}$$

then

$$N_{\mathrm{OUT}} = G\left(N_{\mathrm{IN}} + A\right) \,, \tag{5.77}$$

where

$$\begin{aligned} A &= \left(1 - \frac{1}{G}\right)\left(\mathrm{Var}\left\{X_{2,\mathrm{IN}}^{\theta}\right\} + \mathrm{Var}\left\{X_{2,\mathrm{IN}}^{\theta+\pi/2}\right\}\right) \\ &\leq 2\left(1 - \frac{1}{G}\right) \,. \end{aligned} \tag{5.78}$$

This is in agreement with a general theorem for the noise added by a linear amplifier [7]. The minimum added noise $A = 2(1 - 1/G)$ occurs when $\mathrm{Var}(X_{2,\mathrm{IN}}^{\theta}) = \mathrm{Var}(X_{2,\mathrm{IN}}^{\theta+\pi/2}) = 1$, that is, when the idler is in a coherent or vacuum state.

5.4 Amplitude Squeezed State with Poisson Photon Number Statistics

Finally we consider a simple nonlinear optical model which produces an amplitude squeezed state which has Poissonian photon number statistics [8, 9]. The model describes a quantised field undergoing a self-interaction via the Kerr effect. The Kerr effect is a nonlinear process involving the third-order nonlinear polarisability of a nonlinear medium. The field undergoes an intensity dependent phase shift, and thus we regard the medium as having a refractive index proportional to the intensity of the field.

Quantum mechanically the Kerr effect may be described by the effective Hamiltonian

$$\mathscr{H} = \hbar \frac{\chi}{2} \left(a^{\dagger}\right)^{2} a^{2} , \tag{5.79}$$

where χ is proportional to the third-order nonlinear susceptibility. The Heisenberg equation of motion for the annihilation operator is

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\mathrm{i}\chi a^{\dagger} a\, a . \tag{5.80}$$

As $a^{\dagger}a$, the photon number operator, is a constant of motion the photon number statistics is time invariant. The solution is then

$$a(t) = \mathrm{e}^{-\mathrm{i}\chi t a^{\dagger} a} a(0) . \tag{5.81}$$

Assume the initial state is a coherent state with real amplitude α. The mean amplitude at a later time is then

$$\langle a(\theta) \rangle = \alpha \exp\left[-\alpha^{2}(1-\cos\theta) - \mathrm{i}\alpha^{2}\sin\theta\right] , \tag{5.82}$$

where we have defined $\theta = \chi t$. Typically $\theta \ll 1$ and then

$$\langle a(\theta) \rangle \approx \alpha \mathrm{e}^{-\mathrm{i}\alpha^{2}\theta - \alpha^{2}\theta^{2}/2} . \tag{5.83}$$

This result displays two effects. Firstly, there is a rotation of the mean amplitude by $\alpha^2\theta$; the expected nonlinear phase shift. Secondly, there is a decay of the amplitude which goes quadratically with time. This decay is due to the fact that the Kerr effect transforms intensity fluctuations in the initial coherent state into phase fluctuations (Fig. 5.3). In effect, the initial coherent state error circle undergoes a rotational shearing while the area remains constant.

Inspection of Fig. 5.3 suggests that, at least for short times, this system is likely to produce a squeezed state with reduced amplitude fluctuations. This is indeed the case. For short times $(\theta \ll 1)$ and large intensities $(\alpha^2 \gg 1)$ one finds the minimum variance of the in-phase quadrature approaches the value

$$V(X_1)_{\min} = 0.4 . \tag{5.84}$$

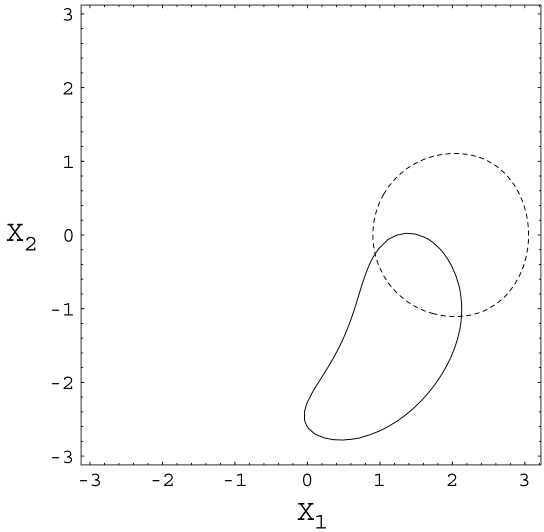

Fig. 5.3 Contour of the Q-function (at a height of 0.3) for the state of a single mode, prepared in a coherent state with $\alpha = 2.0$, evolved with a Kerr nonlinearity for $\theta = 0.25$. The equivalent contour for the initial coherent state is shown as *dashed*

This occurs at the value

$$\theta_{\min} \approx \pm \frac{0.55}{\alpha^2} . \tag{5.85}$$

This short time behaviour is evident in Fig. 5.4a. Thus even though the photon statistics is at all times Poissonian, for short times the field is amplitude squeezed.

We now consider mixing the output of the nonlinear process with a coherent field on a beam splitter of low reflectivity. The output field is now given by

$$a_0 = \sqrt{T} \mathrm{e}^{-\mathrm{i}\theta a^\dagger a} a + \sqrt{R} \beta , \tag{5.86}$$

where β is the coherent amplitude of the mixing field, and T and R are, respectively, the transmitivity and reflectivity of the beam splitter. We assume $T \to 1$ with $\sqrt{R}\beta \to \xi$, that is the mixing field is very strong. In this limit we have

$$a_0 = \mathrm{e}^{-\mathrm{i}\theta a^\dagger a} a + \xi . \tag{5.87}$$

We now can choose ξ so as to minimise the photon number noise at the output. This requires ξ to be $-\pi/2$ out-of-phase with the coherent excitation of the input. As θ increases, the ratio of the number variance to number mean decreases to a minimum at $\theta = 1/2\langle n_0 \rangle^{-2/3}$ (for optimal ξ), and then increases. The minimum photon number variance is [10]

$$V(n_0) = \langle n_0 \rangle^{1/3} , \tag{5.88}$$

where $\langle n_0 \rangle = \langle n \rangle + |\xi|^2$. This is smaller than the similar result for a squeezed state, which has a minimum value of $\langle n \rangle^{2/3}$.

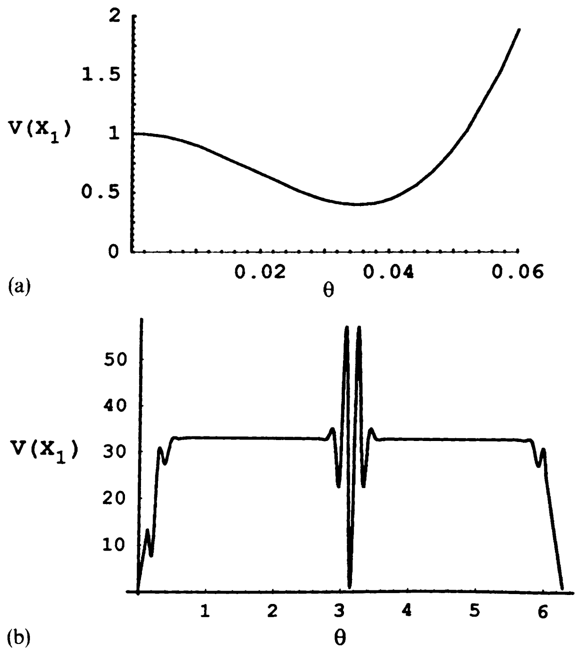

Fig. 5.4 A plot of the variance in the in-phase quadrature for the Kerr interaction, versus the dimensionless interaction time θ, with an initial coherent state of amplitude $\alpha = 4.0$. (**a**) Short time behaviour. (**b**) Long time behaviour

Were the rotational shearing to continue (as one might expect from a classical model) the variance in the in-phase and out-of-phase quadratures would saturate at the value $2\alpha^2 + 1$. This would be the variance for a number state with the photon number equal to α^2. That this does not happen is evident in Fig. 5.4b. Indeed, from (5.82) it is clear that for $\theta = 2\pi$ the mean amplitude returns to the initial value. A similar result holds for the variances (Exercise 5.6). This is an example of a quantum recurrence and arises from the discrete nature of the photon number distribution for a quantised field. The details are left for Exercise 5.6. In fact at $\theta = \pi$ the system evolves to a coherent superposition of coherent states:

$$|\psi(\theta = \pi)\rangle| = \frac{1}{\sqrt{2}}\left(e^{i\pi/4}|i\alpha\rangle + e^{-i\pi/4}|-i\alpha\rangle\right) . \tag{5.89}$$

This phenomenon would be very difficult to observe experimentally as typical values of χ would require absurdly large interaction times, which in practice means extremely large interaction lengths. In Chap. 15 we will show that dissipation also makes the observation of such a coherent superposition state unlikely in a Kerr medium.

Exercises

5.1 Derive the Wigner and P functions for the reduced density operator of the signal mode for the non-degenerate parametric amplifier.

5.2 Show that, $n_1 - n_2$, the difference in the number of photons in the signal and idler mode is a constant for the parametric amplifier.

5.3 The Hamiltonian for the frequency up-converter is

$$\mathscr{H} = \hbar\omega_1 a_1^\dagger a_1 + \hbar\omega_2 a_2^\dagger a_2 + \hbar\kappa\left(\mathrm{e}^{\mathrm{i}\omega t} a_1^\dagger a_2 + \mathrm{e}^{-\mathrm{i}\omega t} a_1 a_2^\dagger\right),$$

where $\omega = \omega_2 - \omega_1$. Show that $n_1 + n_2$, the sum of the number of photons in the signal and idler modes, is a constant.

5.4 Show that the process of parametric frquency upconversion is noiseless, that is a coherent state remains coherent.

5.5 Take the initial state for the frequency upconverter to be $|N,N\rangle$. Express the density operator at time t as the tensor product of number states. *Hint: Use the disentangling theorem*, see (5.63). What is the reduced density operator for a single mode?

5.6 (a) If the initial state for the Kerr-effect model is a coherent state with real mean amplitude, calculate the variances for the in-phase and out-of-phase quadratures. Show that at $\chi t = \pi$ the field exhibits amplitude squeezing for small values of the amplitude.

(b) Show that at $\chi t = \pi$ the state may be written in the form

$$\frac{1}{\sqrt{2}}\left(\mathrm{e}^{\mathrm{i}\pi/4}|-\mathrm{i}\alpha\rangle + \mathrm{e}^{-\mathrm{i}\pi/4}|\mathrm{i}\alpha\rangle\right).$$

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